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The Nahm equations, finite-gap potentials and Lamé functions

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Abstract. It is shown that the Lamé equation, which is the simplest example of a finite-gap Hill's equation, can be written in terms of the composition of two first-order matrix operators, the coefficients of which satisfy Nahm's equation. The characteristic eigenvalues of the Lamé equation emerge from the representation theory of the Lie algebra so(3). A special case is that of reflectionless potentials and their associated bound states.

This paper is concerned with doubly periodic solutions of the Lamé equation

$$\left(\frac{d^2}{dz^2} - n(n+1)k^2 \operatorname{sn}^2 z + h\right) f = 0 \tag{1}$$

where k is the modulus of the elliptic function sn z, and where n and h are constants. If n is an integer, then the operator appearing in (1) is an example of a 'finite-gap' operator, associated with solutions of the periodic Korteweg-de Vries equation (Novikov *et al* 1984).

The basic facts about doubly periodic solutions of (1) are as follows (see Ince 1941, Erdélyi 1955, Arscott 1964). If one wants a solution with real period $2K$ or $4K$, and imaginary period $2iK'$ or $4iK'$, then n must be an integer (without loss of generality, a positive integer), and h must equal one of a set of $2n+1$ characteristic values. These solutions are polynomials ('Lamé polynomials') of degree n in the elliptic functions sn z, cn z, dn z. There are also doubly periodic solutions with periods $8K$ and $8iK'$, if (and only if) n is half an odd integer, and h equals one of a set of $n+\frac{1}{2}$ characteristic values. These 'algebraic Lamé functions' have branch points and are not meromorphic.

We shall see that, for these special values of n and h, the second-order operator in (1) may be factorised into a product of two first-order matrix operators (Dirac operators), the coefficients of which satisfy the so(3) Nahm equations. It will turn out that the characteristic values of n and h are associated with representations of the Lie algebra so(3).

The Nahm equations involve three $N \times N$ matrices $T_1(z)$, $T_2(z)$, $T_3(z)$, with entries that are, in general, complex valued. One can also think in terms of a single $N \times N$ matrix $T(z) = T_j(z)\sigma_j$, taking values in the imaginary quaternions. We think of imaginary quaternions as being generated by the three Pauli matrices σ_j, satisfying

$$\sigma_j \sigma_k = \delta_{jk} + i\varepsilon_{jkl}\sigma_l. \tag{2}$$

(The indices $j, k, l, \ldots$ range over 1, 2, 3 and the Einstein summation convention is used throughout.)

Let Δ and $\tilde{\Delta}$ denote the first-order differential operators

$$\Delta = \frac{\mathrm{d}}{\mathrm{d}z} + T(z)$$

$$\tilde{\Delta} = \frac{\mathrm{d}}{\mathrm{d}z} - T(z)$$

and consider their composition $\tilde{\Delta}\Delta$. Using (2), we obtain

$$\tilde{\Delta}\Delta = \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - T_jT_j\right) + \left(\frac{\mathrm{d}T_l}{\mathrm{d}z} - \mathrm{i}\varepsilon_{jkl}T_jT_k\right)\sigma_l.$$

Thus the operator $\tilde{\Delta}\Delta$ is 'real' (i.e. the part containing the imaginary quaternions σ_l vanishes) if and only if

$$T_l' = \mathrm{i}\varepsilon_{jkl}T_jT_k \tag{3}$$

where the prime denotes $\mathrm{d}/\mathrm{d}z$. These are the Nahm equations.

Note that, in general, T_jT_j is a *complex*-valued $N \times N$ matrix of functions of z (it is 'real' only in the sense that it does not involve the σ_j).

Equations (3) arose in Nahm's construction of non-Abelian monopole solutions; the operator Δ, and the fact that $\tilde{\Delta}\Delta$ is real, are crucial to this construction. For more details, the reader is referred to Nahm (1982, 1986) and Hitchin (1983).

The Nahm equations (3) may, in view of the fact that ε_{jkl} is totally antisymmetric, be rewritten as

$$T_l' = \tfrac{1}{2}\mathrm{i}\varepsilon_{jkl}[T_j, T_k] \tag{4}$$

where [,] denotes the matrix commutator. So it makes sense to think of (4) in Lie algebraic terms, and to regard T_1, T_2 and T_3 as belonging to an N-dimensional irreducible representation of some Lie algebra. To begin with, let us take this Lie algebra to be the simplest (non-trivial) one, namely so(3).

An N-dimensional representation of so(3) is generated by three $N \times N$ matrices t_j satisfying

$$[t_j, t_k] = -\mathrm{i}\varepsilon_{jkl}t_l. \tag{5}$$

So we take each of T_1, T_2 and T_3 to be a linear combination of the t_j (with coefficients which depend on z). The general solution of (4) is then

$$\begin{aligned} T_1 &= -t_1 k \operatorname{sn} z \\ T_2 &= t_2 \mathrm{i} k \operatorname{cn} z \\ T_3 &= t_3 \mathrm{i} \operatorname{dn} z \end{aligned} \tag{6}$$

modulo certain symmetry transformations. (First, we omit the trivial solution in which the T_j all commute and are constant. Second, we use the fact that the so(3) Nahm equations are invariant under the action of two copies of the group SO(3), the first acting by $T_j \mapsto \Lambda_j{}^k T_k$ with $\Lambda \in \mathrm{SO}(3)$ and the second acting on so(3) by the adjoint action. Finally, we use the affine freedom $z \mapsto \alpha z + \beta$.)

From (6) we obtain

$$\begin{aligned} T_jT_j &= t_1^2k^2\operatorname{sn}^2 z - t_2^2k^2\operatorname{cn}^2 z - t_3^2\operatorname{dn}^2 z \\ &= n(n+1)k^2\operatorname{sn}^2 z - (t_3^2 + k^2t_2^2) \end{aligned} \tag{7}$$

where $2n = N - 1$ is the highest weight of the representation, so that t_jt_j equals $n(n+1)$ times the identity $N \times N$ matrix. So the operator $\tilde{\Delta}\Delta$ is

$$\tilde{\Delta}\Delta = \frac{\mathrm{d}^2}{\mathrm{d}z^2} - n(n+1)k^2\operatorname{sn}^2 z + M \tag{8}$$

where $M = M(k)$ is a constant matrix (it depends on k, but not on z):

$$M(k) = t_3^2 + k^2 t_2^2. \tag{9}$$

Thus the equation $\tilde{\Delta}\Delta f = 0$ consists of N copies of Lamé's equation (1), with values of h equal to the eigenvalues of $M(k)$. The doubly periodic solutions we are after are, in fact, solutions of the Dirac equation $\Delta F = 0$.

Note that the matrix M depends on the choice of t_3 and t_2; however, its spectrum depends only on the numbers n and k.

Consider first the simplest integer representation, corresponding to $n = 1$. The t_j are given by

$$t_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad t_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \mathrm{i} & 0 \\ -\mathrm{i} & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{bmatrix} \qquad t_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The matrix $M(k)$ is

$$M(k) = \begin{bmatrix} 1+\frac{1}{2}k^2 & 0 & -\frac{1}{2}k^2 \\ 0 & k^2 & 0 \\ -\frac{1}{2}k^2 & 0 & 1+\frac{1}{2}k^2 \end{bmatrix} \tag{10}$$

the eigenvalues of which are 1, k^2, $1+k^2$. The matrix T appearing in the operator Δ involves the tensor product of the t_j and the Pauli matrices σ_j, and is therefore a 6×6 matrix. It acts on a 6-vector F. Let us write F as a pair of 3-vectors ψ and φ. Then $\Delta F = 0$ becomes

$$\begin{aligned} &\psi' + T_3\psi + (T_1 - \mathrm{i}T_2)\varphi = 0 \\ &\varphi' - T_3\varphi + (T_1 + \mathrm{i}T_2)\psi = 0. \end{aligned} \tag{11}$$

Let $(\psi_1, \psi_0, \psi_{-1})$ denote the components of ψ, and similarly for φ. If we use the expressions (6) for T_j then (11) becomes

$$\begin{aligned} &\sqrt{2}\psi_0' = \mathrm{i}k \exp(-\mathrm{i}\,\mathrm{am}\, z)\varphi_1 - \mathrm{i}k \exp(\mathrm{i}\,\mathrm{am}\, z)\varphi_{-1} \\ &\sqrt{2}\varphi_1' = \mathrm{i}\sqrt{2}(\mathrm{dn}\, z)\varphi_1 + \mathrm{i}k \exp(-\mathrm{i}\,\mathrm{am}\, z)\psi_0 \\ &\sqrt{2}\varphi_{-1}' = -\mathrm{i}\sqrt{2}(\mathrm{dn}\, z)\varphi_{-1} - \mathrm{i}k \exp(\mathrm{i}\,\mathrm{am}\, z)\psi_0 \end{aligned} \tag{12}$$

together with the *same* equations with φ_1 replaced by ψ_{-1}, ψ_0 by φ_0, and φ_{-1} by ψ_1 (Here $\exp(\mathrm{i}\,\mathrm{am}\, z) = \mathrm{cn}\, z + \mathrm{i}\,\mathrm{sn}\, z$.) So the 6-vector equation $\Delta F = 0$ decouples into two identical sets of three equations each.

The general solution of equations (12) depends on three parameters and involves transcendental Lamé functions. But it is easy to exhibit a doubly periodic solution:

$$\psi_0 = \mathrm{dn}\, z \qquad \varphi_1 = -\frac{k}{2\sqrt{2}}\exp(-\mathrm{i}\,\mathrm{am}\, z) \qquad \varphi_{-1} = -\frac{k}{2\sqrt{2}}\exp(\mathrm{i}\,\mathrm{am}\, z). \tag{13}$$

These functions correspond to the $n = 1$ Lamé polynomials. Indeed, the 3-vector $f = \text{column } (\varphi_1, \psi_0, \varphi_{-1})$ is clearly a solution of

$$f'' - n(n+1)k^2(\mathrm{sn}^2\, z)f + Mf = 0 \tag{14}$$

and if we diagonalise the matrix M, then the components of the transformed f will be the three $n = 1$ Lamé polynomials, corresponding to the three characteristic values $h = 1$, k^2, $1 + k^2$.

It is easy to extend the above to $n=2,3,4,\ldots$. The equations $\Delta F=0$ decouple into two copies of a set of $N=2n+1$ equations and this set has a solution of the form

$$f=\text{column}(\varphi_n,\psi_{n-1},\varphi_{n-2},\ldots,\psi_{1-n},\varphi_{-n})$$

where

$$\varphi_k=\alpha_k\exp(-\mathrm{i}k\,\mathrm{am}\,z)\qquad \psi_k=\alpha_k\,\mathrm{dn}\,z\exp(-\mathrm{i}k\,\mathrm{am}\,z).$$

Here $\alpha_n,\alpha_{n-1},\ldots,\alpha_{-n}$ are non-zero constants: it is straightforward to compute their values, which are determined uniquely up to an overall constant of proportionality, in terms of the entries in the matrices t_j generating the N-dimensional representation.

So as before, f satisfies (14). The N components of f are clearly linearly independent over $\mathbb{C}$, so that, after diagonalising M, we obtain N independent solutions of the Lamé equation, one for each eigenvalue h of M. One knows (Ince 1941) that no two Lamé polynomials can belong to the same value of h; so the eigenvalues of M are all distinct and give all of the N characteristic values h.

The case of half-integer (i.e. even-dimensional) representations of so(3) is slightly different. For example, in the case $n=\frac{1}{2}$, the matrix $M(k)$ has two *equal* eigenvalues $\frac{1}{4}(1+k^2)$ and the *general* solution of $\Delta F=0$ can be written down explicitly: each component of F is a linear combination of the two functions $(\mathrm{dn}\,z+\mathrm{cn}\,z)^{1/2}$ and $(\mathrm{dn}\,z-\mathrm{cn}\,z)^{1/2}$. These are the two algebraic Lamé functions corresponding to $n=\frac{1}{2}$.

Let us examine what happens in the limit $k=0$. If we first make the affine transformation $z\mapsto \mathrm{i}z+K+\mathrm{i}K'$ and then let k tend to zero, the equation $\tilde{\Delta}\Delta f=0$ becomes

$$f''+[n(n+1)\,\mathrm{sech}^2\,z-t_3^2]f=0. \tag{15}$$

The potential appearing in (15) is, of course, well known as a reflectionless potential, provided n is an integer (Lamb 1980, § 2.5). The non-zero eigenvalues of t_3^2 correspond to the bound states of the Schrödinger equation (15). The solution (13) of $\Delta F=0$ becomes

$$(\varphi_1,\psi_0,\varphi_{-1})=(\mathrm{i}2^{-1/2}\,\mathrm{sech}\,z,\,-\tanh z,\,0).$$

So φ_1 is the normalisable solution of (15) corresponding to the eigenvalue 1, whereas ψ_0, corresponding to the eigenvalue 0, is not normalisable.

Finally, one can ask whether it is possible to generalise the above observations, so as to obtain more general finite-gap and reflectionless potentials, together with their associated eigenvalues. The first guess would be to use a Lie algebra g other than so(3), so that T_j takes values in a representation of g. One knows that, for any g, the g-Nahm equations are completely integrable and can be solved in terms of Abelian functions and finite-gap potentials can also be expressed in terms of Abelian functions, so at first sight it seems promising. However, it does not work, as the following argument shows.

Let G_α, $\alpha=1,2,\ldots,m$, be a basis for g. So we can write $T_j=T_j^\alpha(z)G_\alpha$. We want

$$T_j^\alpha T_j^\beta=\xi(z)g^{\alpha\beta}+M^{\alpha\beta} \tag{16}$$

where $g^{\alpha\beta}$ is the Killing metric and $M^{\alpha\beta}$ is constant. For then we would have

$$T_jT_j=\xi(z)C+M$$

where M is a constant matrix and C (the Casimir element) is a scalar (compare equation (7)). However, the left-hand side of (16), thought of as an $m\times m$ matrix, has rank $\leqslant 3$. So the right-hand side cannot have the required form, unless either ξ is a constant or $m=3$.

It remains a possibility that some generalisation of the Nahm–Dirac operator Δ could provide the general finite-gap potentials and their eigenvalues. This is worth investigating further.

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References

Arscott F M 1964 *Periodic Differential Equations* (Oxford: Pergamon)

Erdélyi A 1955 *Higher Transcendental Functions* (*Bateman Project*) vol 3 (New York: McGraw-Hill)

Hitchin N J 1983 *Commun. Math. Phys.* **89** 145

Ince E L 1941 *Proc. R. Soc. Edinburgh* **60** 47, 83

Lamb G L 1980 *Elements of Soliton Theory* (New York: Wiley)

Nahm W 1982 *Group Theoretical Methods in Physics* (*Lecture Notes in Phys.* **180**) ed M Serdaroglu and E Inonu (Berlin: Springer) p 456

—— 1986 *Non-Linear Equations in Classical and Quantum Field Theory* (*Lecture Notes in Phys.* **226**) ed N Sanchez (Berlin: Springer)

Novikov S, Manakov S V, Pitaevskii L P and Zakharov V E 1984 *Theory of Solitons* (New York: Consultants Bureau)